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CONVERGENT LAMINAR FLUID FLOW BETWEEN TWO ROTATING DISKS

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A stationary flow from the periphery to the center in a hollow between two coaxial, closely located rotating disks is studied by the iterative method of solving a system of equations of the dynamics of a viscous incompressible fluid. The existence and uniqueness of the approximate solution are shown.

The specific feature of the geometrical structure of hollows between two rotating disks or cones in various rotor machines (separators, centrifuges, and total-head disks) is a comparatively small (up to 0.01) ratio of the channel height to the hollow radius. In studying the flow in the interdisk channel within the framework of the assumptions of a boundary layer, this circumstance allows one to simplify the initial set of equations of the dynamics of a viscous incompressible fluid, as is done for a flow in a slit channel.

The specific features of convergent fluid flows between two rotating, closely located disks and cones are analyzed in [1–5], where one algorithm or another is used as a method of quantitative analysis of the equations of motion, the velocity and pressure fields are calculated in detail, and convergence of the solutions is estimated. However, the problems of correctness of the approximate solutions obtained were omitted in these studies.

It is convenient to analyze the specific features of a fluid flow between two disks of radius r_0 that rotate with the same or different angular velocities with the use of the cylindrical coordinate system (r, θ, and z) rigidly connected to one of the disks. For definiteness, without loss of generality of the formulation of the problem, we consider that, together with the fluid, the disks are suddenly induced into rotation with a constant angular velocity ω; the fluid enters a narrow gap between the disks at the distance r_0 over the entire circumference and is then removed through a round orifice at the distance $r_1 < r_0$ (Fig. 1). Let u and v and w be the radial and relative circumferential and transverse components of the fluid velocities, respectively, p be the pressure, ρ be the density, and ν be the kinematic-viscosity coefficient. Locating the coordinate origin on the rotation axis in the middle between the disks, one can adopt the conditions of fluid adhesion to the channel walls as the boundary conditions:

$$u = v = w = 0 \quad \text{for} \quad z = \pm h/2, \qquad r_1 < r < r_0, \tag{1}$$

where h is the distance between the disks. Because the fluid moves toward a hollow between the disks from the periphery at a pressure exceeding the discharge pressure, we adopt the additional boundary condition

$$p(r_1, z) = \omega^2 r_1^2/2. \tag{2}$$

Making a comparative estimate of the viscous terms in the Navier–Stokes equations, for the flow considered, in the selected coordinate system we obtain approximately the system

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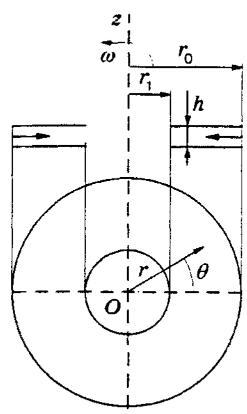

Fig. 1. Scheme of fluid motion between two rotating disks.

$$\frac{\nu\,\partial^2 u}{\partial z^2} = U + \frac{\partial \Phi}{\partial r}; \tag{3}$$

$$\frac{\nu\,\partial^2 v}{\partial z^2} = V; \tag{4}$$

$$\frac{\nu\,\partial^2 w}{\partial z^2} = W + \frac{\partial \Phi}{\partial z}; \tag{5}$$

$$\frac{\partial (ru)}{\partial r} + \frac{r\,\partial w}{\partial z} = 0. \tag{6}$$

Here

$$U = \frac{u\,\partial u}{\partial r} + \frac{w\,\partial u}{\partial z} - \frac{v^2}{r} - 2\omega v, \qquad V = \frac{u}{r}\frac{\partial (rv)}{\partial r} + \frac{w\,\partial v}{\partial z} + 2\omega u,$$
$$W = \frac{u\,\partial w}{\partial r} + \frac{w\,\partial w}{\partial z}, \qquad \Phi = \frac{p}{\rho} - \frac{\omega^2 r^2}{2}. \tag{7}$$

By virtue of the symmetry of the velocity field with respect to the plane $z = 0$, we choose

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0, \qquad w = 0 \quad \text{for} \quad z = 0; \tag{8}$$

$$u = v = w = 0 \quad \text{for} \quad z = h/2 \tag{9}$$

instead of the boundary conditions (1).

In addition, if l is the characteristic longitudinal dimension of the disk (for example, $l = r_0$), the transverse component of the flow velocity is $w = O(uh/l)$ according to (6). Then, comparing Eqs. (3) and (5), we conclude that if $h/l \ll 1$, we have $(\partial\Phi/\partial z)/(\partial\Phi/\partial r) \ll 1$; consequently, we have approximately $\Phi = \Phi(r)$ for the dynamic pressure. Hereinafter, this circumstance is effectively used to show that the iterative procedure of the asymptotic solution of system (3)–(6), which is in agreement with (2), (8), and (9), is correct.

We now pass to the dimensionless quantities

$$r = lr', \qquad z = lx, \qquad u = V_* u', \qquad v = V_* v', \qquad w = V_* w',$$

$$\Phi = V_*^2 \Phi', \qquad l = (\nu/\omega)^{1/2}, \qquad V_* = (\nu\omega)^{1/2}.$$

Omitting the primes, we rewrite Eqs. (3) and (4) in dimensionless form

$$\frac{\partial^2 u}{\partial x^2} = U + \frac{\partial \Phi}{\partial r}, \qquad \frac{\partial^2 v}{\partial x^2} = V. \tag{10}$$

The boundary conditions (8) and (9) can also be written in dimensionless form

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0, \quad w = 0 \quad \text{for} \quad x = 0,$$
$$u = v = w = 0 \quad \text{for} \quad x = \mu \quad (\mu = h(\omega/\nu)^{1/2}/2), \tag{11}$$

where μ is the Ekman number. In addition, the conditions

$$r \int_0^{\mu} u(r, x)\, dx = -q_1, \qquad \Phi(r_1, x) = 0 \tag{12}$$

should be satisfied [$q_1 = QV_*^{-1} l^{-2}/(4\pi)$, where Q is the fluid flow rate].

To construct an iterative algorithm of the solution of the problem, we integrate Eqs. (10) twice over x with allowance for (11) and (12):

$$u(r, x) = -\int_x^{\mu}\int_0^{x} U\, dx_1\, dx + u^0, \qquad v(r, x) = -\int_x^{\mu}\int_0^{x} V\, dx_1\, dx, \tag{13}$$

where $u^0 = 0.5\Phi'(x^2 - \mu^2)$.

According to (12) and (13), the dynamic pressure is determined as a solution of the integral equation

$$\int_0^{\mu}\int_x^{\mu}\int_0^{x} [U + \Phi'(r)]\, dx_2\, dx_1\, dx = -q_1/r;$$

whence

$$\Phi'(r) = \frac{3}{\mu^3}\left(\frac{q_1}{r} - \int_0^{\mu}\int_x^{\mu}\int_0^{x} U\, dx_2\, dx_1\, dx\right). \tag{14}$$

The transverse velocity w is found from the continuity equation (6):

$$w = -\frac{1}{r}\frac{\partial}{\partial r}\int_0^{x} ur\, dx;$$

we note that, according to (12), the boundary conditions (11) for w are satisfied.

Because the velocity field, the pressure, and the boundary conditions for the velocity and pressure components at the entrance of the gap are found from Eqs. (13) and (14) in this formulation of the problem, its solution has an asymptotic character.

According to the iteration method and formulas (13) and (14), for the nth and $(n+1)$th approximations we have

$$\Phi_n'(r) = \frac{3}{\mu^3}\left(\frac{q_1}{r} - \int_0^{\mu}\int_x^{\mu}\int_0^{x} U_n\, dx_2\, dx_1\, dx\right), \qquad u_{n+1} = u_n^0 - \int_x^{\mu}\int_0^{x} U_n\, dx_1\, dx,$$
$$v_{n+1} = -\int_x^{\mu}\int_0^{x} V_n\, dx_1\, dx, \qquad w_n = -\frac{1}{r}\frac{\partial}{\partial r}\int_0^{x} u_n r\, dx, \tag{15}$$

where $U_n = u_n\,\partial u_n/\partial r + w_n\,\partial u_n/\partial x - v_n^2 - 2v_n$, $V_n = u_n\,\partial v_n/\partial r + u_n v_n/r + w_n\,\partial v_n/\partial x + 2u_n$, $u_n^0 = \Phi_n'(x^2 - \mu^2)/2$, and $U_0 = V_0 = 0$ ($n = 0, 1, \ldots$).

If the laminar flow is assumed to move as a quasirigid body in the zeroth approximation, we obtain

$$\Phi_0' = 3q_1/(\mu^3 r) \tag{16}$$

according to (15). In correspondence with (15) and (16), we have

$$\Phi_1' = (3/\mu^3)[q_1/r + 6q_1^2/(35r^3)], \quad u_1 = u^0, \quad v_1 = 0, \quad w_1 = 0$$

in the first approximation and

$$u_2 = [3q_1/(2\mu^3 r)](x^2 - \mu^2), \quad v_2 = [-q_1/(4\mu^3 r)](x^2 - \mu^2)(5\mu^2 - x^2), \quad w_2 = 0$$

in the second approximation, i.e., according to the adopted scheme, the nonzero values of the circumferential and transverse flow velocities are determined, beginning with the second and third stages of iterations, respectively.

Thus, with one-place accuracy, we arrive at the calculation formulas for a divergent flow of a fluid obtained by the method of expanding the solution as series in inverse powers r for a divergent laminar (small values of μ and q_1) flow [6]. Substituting the found values of u and v into the right sides of relations (15), we find the third approximation, which is omitted here because if its cumbersome form. In the general form, the structure of the calculation formulas of flow kinematics of in the nth approximation (n partial sums) has the form

$$u_n(r,x) = \sum_{k=1}^{n} \frac{q^k u_{nk}(x)}{r^{2k-1}}, \quad v_n(r,x) = \sum_{k=1}^{n} \frac{q^k v_{nk}(x)}{r^{2k-1}}, \quad w = 2\sum_{k=1}^{n}(k-1)q^k \int_0^x \frac{u_{nk}}{r^{2k}}\,dx, \tag{17}$$

i.e., the partial sums are the functional series that are parametric relative to r and power relative to x. Determination of the fourth- or higher-order iterations in an explicit form is a quite complicated problem. However, higher-order approximations polynomial in x can be found with the use of modern languages of symbol programming as a result of an explicit multiple integration; this makes it possible to estimate both the convergence and the speed of convergence of the resulting iterative solution.

The calculation results for the longitudinal component of the flow velocity over the three approximations with one-place accuracy are in qualitative and quantitative agreement with the data for the divergent flow regime (Fig. 2).

In studying the convergence of the iterative calculation procedure, we confine ourselves to the case of a slow quasicircular flow regime, i.e., we assume that the parameters μ and q_1 are not large. With allowance for the fact that the nonlinear terms in expressions (7) for U, V, and W are the quantities of the same order of smallness, to simplify the substantiation of the convergence of iterations, we retain in (7) only convective terms that contain the velocity products. Then, we obtain approximately

$$u_{n+1} = -\int_x^\mu \int_0^x U_n\,dx_1\,dx + \frac{3(\mu^2 - x^2)}{2\mu^3} \int_0^\mu \int_x^\mu \int_0^x U_n\,dx_2\,dx_1\,dx + u^0,$$

$$v_{n+1} = -\int_x^\mu \int_0^x V_n\,dx_1\,dx, \tag{18}$$

where $U_n = -2v_n - v_n^2/r$, $V_n = 2u_n + u_n v_n/r$, $u^0 = q(x^2 - \mu^2)/r$, and $q = 1.5q_1/\mu^3$ is the Rossby number.

In contrast to [7], where the stationary divergent flow regime for a fluid between two infinite disks is investigated, for the flow-velocity field, not only the centripetal v^2/r but also the rotary uv/r acceleration is taken into account in the computational scheme (18).

Hereinafter, in constructing the majorant for the partial sums (17), we take into account, that, for a slow flow regime (small Ekman and Rossby numbers), according to (3) and (4) the maximum absolute values of the longitudinal and circumferential fluid-velocity components are reached in the middle of a slit ($x = 0$). By virtue of (18), we have

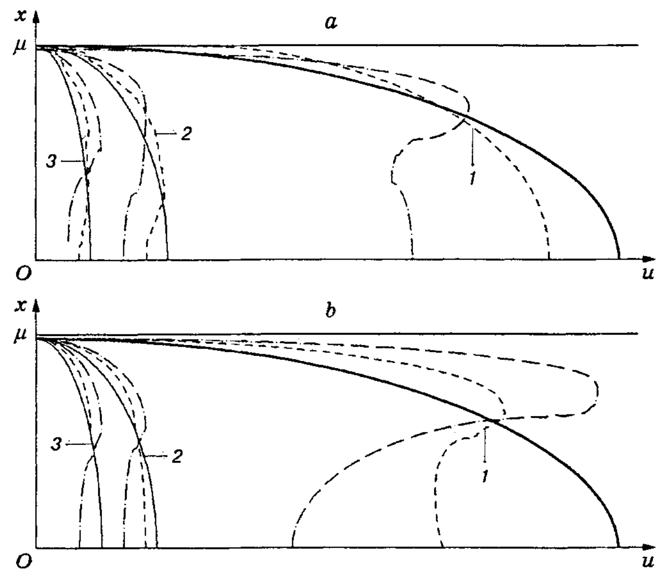

Fig. 2. Structures of the longitudinal component of the flow velocity for $r_0' = r_0/h = 100$: curves 1 refer to $r = 0.1r_0'$ and $\mu = 1.12$, curves 2 to $r = 0.5r_0'$ and $\mu = 1.58$, and curves 3 to $r = 0.9r_0'$ and $\mu = 1.69$; solid curves refer to $q = 7.04$ (a) and 35.12 (b), dashed curves to $q = 9.96$ (a) and 49.80 (b), and dot-and-dashed curves to $q = 12.18$ (a) and 60.92 (b).

$$u_{n+1} = -\int_0^\mu\int_0^x U_n\,dx_1\,dx - \frac{1.5}{\mu}\int_0^\mu\int_x^\mu\int_0^x U_n\,dx_2\,dx_1\,dx - \frac{q\mu^2}{r}, \qquad v_{n+1} = -\int_0^\mu\int_0^x V_n\,dx_1\,dx. \tag{19}$$

Hereinafter, in the expressions for U_n and V_n, we assume that $u_n = u_n(0)$, and $v_n = v_n(0)$ ($n = 0, 1, \ldots$). Here, owing to (18) and (19), the inequalities

$$|u_{n+1}| \leqslant \mu^2[2.5(2|v_n| + v_n^2/r) + q/r], \qquad |v_{n+1}| \leqslant \mu^2(2|u_n| + |u_n v_n|/r)$$

hold. Omitting the signs of the modulus and amplifying the inequalities, we obtain

$$u_{n+1} < 5\mu^2(v_n + v_n^2/r + q/r), \qquad v_{n+1} < 2\mu^2(u_n + u_n v_n/r).$$

Further amplified, the inequalities take the form

$$u_{n+1} < \eta(q/r + v_n + v_n^2/r), \qquad v_{n+1} < \eta(u_n + u_n v_n/r), \qquad \eta = 5\mu^2. \tag{20}$$

Here assuming that $v_0 = 0$ and $u_0 < \eta q/r$, for $n \geqslant 0$ we have

$$u_1 < q\eta/r, \quad v_1 < q\eta^2/r, \quad u_2 < (q\eta/r)(1 + \eta^2 + q\eta^4/r^2), \quad v_2 < (q\eta^2/r)(1 + q\eta^2/r^2),$$

$$\begin{aligned} u_3 &< (q\eta/r)[1 + \eta^2 + q\eta^2(1 + 2\eta^2)/r^2 + q^2\eta^4(1 + 2\eta^2)/r^4 + q^3\eta^8/r^6], \\ v_3 &< (q\eta^2/r)[1 + \eta^2 + q\eta^2(1 + 2\eta^2)/r^2 + q^2\eta^4(1 + 2\eta^2)/r^4 + q^3\eta^8/r^6], \\ u_4 &< (q\eta/r)[\eta(1 + \eta^2) + q\eta^3(2 + 4\eta^2 + \eta^4)/r^2 + q^2\eta^5(1 + 2\eta^2)(3 + 2\eta^2)/r^4 \\ &\quad + q^3\eta^7(3 + 10\eta^2 + 8\eta^4)/r^6 + 2q^4\eta^9(1 + 5\eta^2 + 5\eta^4)/r^8 \\ &\quad + q^5\eta^{11}(1 + 2\eta^2)(1 + 4\eta^2)/r^{10} + 2q^6\eta^{15}(1 + 2\eta^2)/r^{12} + q^7\eta^{19}/r^{14}], \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \text{ etc.} \end{aligned} \tag{21}$$

In the adopted assumptions, one can consider that $q\eta^2/r^2 < 1$, where $\eta < 1$.

As follows from formulas (20) and the analysis of the structure of the coefficients on the right side of (21), the coefficients $a_k(\eta)$ and $a_{k+1}(\eta)$ for $(q\eta^2/r^2)^k$ and $(q\eta^2/r^2)^{k+1}$ do not depend explicitly on the index k and are magnitudes of the same order; we note that these coefficients first grow monotonically, reaching the greatest value of a_m ($m \approx 2^{n-1}$, where n is the order of iteration) and then decrease. It is clear that $r^2/(q\eta^2)$ can be selected so large that $r^2/(q\eta^2) > \lim(a_{k+1}/a_k)$ as $r \to \infty$, $k \to \infty$, and $n \to \infty$. Whence, we have the convergence of the sequences $u_n(0)$ and $v_n(0)$ as $n \to \infty$ and, hence, the uniform convergence of the partial sums $u_n(x)$, and $v_n(x)$ for $x \in [0,\mu]$ to the continuous solution of the boundary-value problem formulated in the range of values of $q\eta^2/r^2 < 1$ and $\eta < 1$, where $q = 6Q/(\pi\omega h^3)$ and $\eta = 5\mu^2$.

Thus, within the framework of the formulation of the problem the value of the Ekman criterion $\eta < 1$ that is common in the flow region and the value of the Rossby criterion $q < (r/\eta)^2$ at each point of the flow is a sufficient convergence condition for its solution.

Since from the proof of convergence of the iterative procedure follows the correctness (for fixed values of the parameters r, q, and η) and $C[0,\mu]$) of the norms $\|v\|$ and $\|u\|$ in the metric space of continuous functions on the segment $[0,\mu]$, which are defined in the form [8]

$$\|u\| = \max|u(x)|, \qquad \|v\| = \max|v(x)|, \quad x \in [0,\mu],$$

the resulting iterative solution converges in norm in the space $C[0,\mu]$. With a correct uniform convergence of the expansion of the solution of the problem, for example, for $u(r,x)$, it follows that this solution is unique. Let $u_0 = u_*^0$ and $v_0 = v_*^0$ be the initial distribution of the flow velocities (u_*^0 and v_*^0 are functions that have the same properties as u^0). By virtue of (13) and (15), we have

$$u_{n+1} = -\int_x^\mu\int_0^x U_n(u^0,u_*^0,v_*^0)\,dx_1\,dx + \int_0^\mu\int_x^\mu\int_0^x U_n(u^0,u_*^0,v_*^0)\,dx_2\,dx_1\,dx + u^0,$$

$$v_{n+1} = -\int_x^\mu\int_0^x V_n(u^0,u_*^0,v_*^0)\,dx_1\,dx, \qquad n \geqslant 0. \tag{22}$$

Then, under the assumption that u^0, u_*^0, and v_*^0 belong to the space $C[0,\mu]$, similarly to (21), in the field of the same flow parameters one can construct the majorants for u and v in the case where the flow-velocity profiles at the entrance of the channel have the form $u_0 = u_*^0$ and $v_0 = v_*^0$. Since u and v are the uniformly convergent series, for the longitudinal component of the flow rate one can write

$$u_{n+1} = \sigma_n' + \sigma_n'', \tag{23}$$

where σ_n' depends only on u^0, and σ_n'' on u^0 and u_*^0 and v_*^0 (or only on u_*^0 and v_*^0).

Let the sequence of partial sums $u_1, u_2, \ldots$ defined in a certain domain $S(r,x)$ uniformly converges on the set S. According to the Cauchy criterion of uniform convergence of the sequence, this means that, for an arbitrary $\varepsilon > 0$, there is a number ν that does not depend on $(r,\ x)$, such that it follows that $|u_k(r,x) - u_l(r,x)| < \varepsilon$ from $k > \nu$ and $l > \nu$ for all $(r,x) \in S$ [8]. Here, since the number n of the partial sum [the order of approximation (17)] can always be selected so large that $n > \nu$, k, and l [then, $|u_k - u_l|$ would depend only on u^0 by virtue of (23)], then the solution of the problem that results from the transition to the limit as $n \to \infty$ in (22) is unique.

In conclusion, it should be mentioned that the results of the theoretical analysis of a fluid flow in a narrow gap between two disks that rotate with the same velocity when the fluid is supplied from the periphery were compared by R. Adams, W. Rice, and other authors (see, e.g., [1, 2, 4]) with the experimental data; for this purpose, the pressure was measured along the flow between the disks. Satisfactory agreement between the experimental and the theoretical results has been obtained in the range of variation of the flow parameters, which shows the correctness of the chosen model in the studied range of flow parameters.

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